AN EXPLORATION OF SYNTHETIC DIVISION FOR NON-LINEAR POLYNOMIAL DIVISORS

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ABSTRACT

As a goal of developing alternative algorithm on division of polynomials whose dividend is \( a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \cdots + a_nx + a_{n+1} \) and the divisor is \( b_1x^m + b_2x^{m-1} + b_3x^{m-2} + \cdots + b_mx + b_{m+1} \), where \( n > m \), \( a_1 \neq 0 \), \( b_1 \neq 0 \), and \( a_i \)'s and \( b_i \)'s including \( a_{n+1} \) and \( b_{m+1} \) are constant, the researcher explored the synthetic division in compact form. The researcher believes that the algorithm in this study is a good alternative in dividing polynomials of higher degrees. In the said compact form, division of only some distinct pairs of non-linear polynomials were presented and served as reference problems for the researcher during the initial exploration. The Division Algorithm for Polynomials theorem was applied in exploring the problems with quadratic, cubic and quartic divisors. This resulted to the development of formulas for the coefficients \( t_i \) of the quotient \( Q(x) \) and coefficients \( r_i \) of the remainder \( R(x) \) which were considered important parts of the algorithm. Aside from the condition for the inapplicability for non-linear divisors, additional conditions were provided to the problems where the usual synthetic division is inappropriate. At the end, the algorithm on division of polynomials of higher degrees by non-linear divisors was developed using basic research. Illustrative examples of dividing polynomials using the developed algorithms with monic (\( b_1 = 1 \)) and non-monic (\( b_1 \neq 1 \)) divisors were provided. Results were verified through other existing methods: long division and synthetic division in compact form.

Keywords: synthetic division, high school algebra, short division

INTRODUCTION

The division of polynomials with rational coefficients (usually integers) is one of the lessons in Algebra undertaken by high school students. The predicament lies with the fact that that most textbooks used to teach this topic contains the same algorithm, an algorithm that mimics the long division algorithm taught in Elementary level. There are instances where this long division is not practical to use since the it is tedious and requires more time before one finally gets the quotient. Also, the process becomes more complicated since variables are involved in all the operations used in long division. To minimize the difficulties encountered here, a more efficient algorithm called synthetic division has been introduced. However, it is just usually applied to problems with linear divisor.

There are challenges that surfaced relative to this method. Larson, Hostetller and Edwards (1997) claimed that synthetic division is good only if the divisor is of the form \( x - k \). Accordingly, this method cannot be applied by quadratic divisor like \( x^2 - 3 \). And it is the actual practice as observed in the implementation of our present curriculum. Relatively, Y. L. Cheung (1990) noted the ordinary synthetic division as an abridged version of division of polynomials of any degree by a linear polynomial. In fact, he showed three from among several versions generated after asking about hundred mathematics teachers to generalize this operation.

As a development, new forms of synthetic division, such as extended synthetic division have been introduced to avoid the tedious traditional division of polynomials. Lianghuo Fan (2003) presented a traditional synthetic division method.
On the other hand, Li Zhou (2009) justified why and how the algorithm on synthetic division worked. From here, he presented different forms or arrangements of arrays and successive products.

The first was the diagonal form. Since this form is not space-efficient, it was translated to horizontal form, then switched to compact form. To avoid the issue of knowing in advance the number of rows needed between the starting and the answer rows, the products are written above the starting row and converted the compact arrangement with respect to natural gravity piling, thus this synthetic division was called Short Division of polynomials.

K. L. Ang (2012) in his examples stick to the synthetic division of polynomials by a quadratic divisor. He also claimed that the synthetic division can easily be expanded for a divisor of any degree of polynomials.

Review of the related research and studies also contributed significantly to highlight the intent of this research. There are beliefs and perceptions that Mathematics is a challenging and difficult subject. This is supported by Gafoor and Kurukkan (2015) in the findings of their study that most students considered it as a difficult subject due to adverse teaching style; difficulties in following the instruction, understanding the subject, and remembering the equations and ways to solve problem.

According to Syahputra and Suhartini (2014), the difficulties in mathematics affected the lower students’ achievement. To address the issue of ways to solve problems, the teacher is expected to look for alternative methods. Machisi, et al. (2013) recommends that students should be offered opportunities to try out a variety of mathematical solution strategies rather than confine them to only strategies in their prescribed textbooks. Also, Syahputra and Suhartini (2014) said that a teacher needs an ability to design and implement a variety of learning methods that are considered to match the interests and talents and in accordance with the development of the students.

As regards synthetic division, related articles have been published and support studies have been conducted as well. Machisi et al (2013) underscored that on average, students achieved better scores with the synthetic division strategy than with long division and equating coefficients strategies. Bales (2014) presented a deeper appreciation of synthetic division. He remarkably showed the application of synthetic division in the “evaluation of complex functions” which he claimed to be much easier than DeMoivre’s theorem computation.

More surprisingly, S. Subha, an Indian Information Technology expert proposed an algorithm model to perform synthetic division simulated with Quartus2 toolkit. With his proposal, the algorithm performance is less when compared with restoring and non-restoring division algorithms and it was observed that there was a timing improvement of 29.026%. These challenges and development thus inspired the researcher to do an exploration on synthetic division of polynomials with non-linear divisors, to delve deeper on the process and to develop an alternative, yet specific algorithm that can practically be used by teachers and students. Thus, the main goal of this research was to develop an alternative algorithm on division of polynomials whose dividend is $a_1 x^n + a_2 x^{n-1} + a_3 x^{n-2} + \cdots + a_n x + a_{n+1}$ and the divisor is $b_1 x^m + b_2 x^{m-1} + b_3 x^{m-2} + \cdots + b_m x + b_{m+1}$, where $n > m$, $a_1 \neq 0$, $b_1 \neq 0$, and $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m$ including $a_{n+1}$ and $b_{m+1}$ are constant.

**OBJECTIVES OF THE STUDY**

This study aimed to develop an algorithm using synthetic division in compact arrangement when:

1. $b_1 = 1$
2. $b_1 \neq 1$

It sought to provide additional conditions to the problems where the usual synthetic divisions are inappropriate. Lastly, it also designed to simplify the developed algorithm of dividing polynomials of higher degrees.

**MATERIALS AND METHODS**

As to its intent, this study utilized basic research to arrive at algorithms or formulas for the quotient of two polynomials with non-linear divisors. Basic research, which is also known as
pure or fundamental research, is a way of generating new ideas, principles and theories. On the goals of basic research, Olesen (2017) believed that using the knowledge for anything concrete is not important but instead, as long as we improve our understanding. On the other hand, Nissen (2017) said something relative to good basic research. It can be concluded that all good basic researches eventually have practical applications that lead to new perspectives and projects. With these, basic research is believed to be the most appropriate method to realize the concerns and goals of this paper. It also does not employ any statistical treatment.

As basic research, the starting point was the revisit and review of the Short Division of Polynomials introduced by Li Zhou (2009) which was developed from rearranged elements (addends) of compact synthetic division. Following the design and intent of this study, the researcher applied the Division Algorithm for Polynomials in deriving and verifying the formulas of the needed coefficients of both quotient and remainder. Procedure on dividing any polynomial by quadratic divisor, cubic divisor, and quartic divisor for both \( b_1 = 1 \) and \( b_1 \neq 1 \) was then presented as initial result. From the illustrative examples provided, the developed procedure or algorithm was verified to have the same results when compared with that of already-known and already-established methods of dividing polynomials, the long method and Li Zhou’s short division. To see the further applicability of this algorithm as well, additional conditions for problems where usual or ordinary synthetic divisions are inappropriate. In the end, generalized formulas for the coefficients of the quotient and of the remainder were presented.

RESULTS AND DISCUSSION

Developed Algorithm

The algorithm is hereby presented to determine the quotient of two polynomials whose dividend is \( a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \cdots + a_nx + a_{n+1} \) and the divisor is \( b_1x^m + b_2x^{m-1} + b_3x^{m-2} \cdots + b_mx + b_{m+1} \)

When \( b_1 = 1 \)

1. Use \( n \), the \( \text{deg}(P) \) and \( m \), the \( \text{deg}(D) \) to set the equations of the quotient \( Q(x) \) and the remainder \( R(x) \). To do this,
   a. if \( m = 2 \), then
      \( Q(x) = q_1x^{n-2} + q_2x^{n-3} + q_3x^{n-4} + \cdots + q_{n-2}x + q_{n-1} \) and
      \( R(x) = r_1x + r_2 \)
   b. if \( m = 3 \), then
      \( Q(x) = q_1x^{n-3} + q_2x^{n-4} + q_3x^{n-5} + \cdots + q_{n-3}x + q_{n-2} \) and
      \( R(x) = r_1x^2 + r_2x + r_3 \)
   c. if \( m = 4 \), then
      \( Q(x) = q_1x^{n-4} + q_2x^{n-5} + q_3x^{n-6} + \cdots + q_{n-4}x + q_{n-5} \) and
      \( R(x) = r_1x^3 + r_2x^2 + r_3x + r_4 \)

2. Refer from \( P(x) \) and \( D(x) \) and list the values of \( a_1, a_2, \ldots, a_{n+1} \) and \( b_1, b_2, \ldots, b_{m+1} \) respectively.

3. Evaluate the coefficients of \( Q(x) \) and \( R(x) \) by applying the respective formulas for \( t_i \) and \( r_i \), and substituting the values of \( a_1, a_2, \ldots, a_{n+1} \) and \( b_1, b_2, \ldots, b_{m+1} \). That is, in the division of any polynomial
   a. by a quadratic divisor,
      \( q_n = a_n - b_2q_{n-1} - b_3q_{n-2} \) where \( n \in \mathbb{Z}^+ \)
      while
      \( r_1 = a_n - b_2q_{n-1} - b_3q_{n-2} \)
      \( r_2 = a_{n+1} - b_3q_{n-1} \)
      where particularly for \( r_1 \) and \( r_2 \), \( n \) is the \( \text{deg}(P) \);
   b. by a cubic divisor,
      \( q_n = a_n - b_2q_{n-1} - b_3q_{n-2} - b_4q_{n-3} \) where \( n \in \mathbb{Z}^+ \)
      while
      \( r_1 = a_n - b_2q_{n-1} - b_3q_{n-2} - b_4q_{n-3} \)
      \( r_2 = a_{n+1} - b_3q_{n-1} - b_4q_{n-2} \)
      \( r_3 = a_{n+2} - b_4q_{n-2} \)
      where particularly for \( r_1, r_2 \) and \( r_3 \), \( n \) is the \( \text{deg}(P) \); and
   c. by a quartic divisor,
\[ q_n = a_n - b_2 q_{n-1} - b_3 q_{n-2} - b_4 q_{n-3} - b_5 q_{n-4} \text{ where } n \in \mathbb{Z}^+ \]

while

\[ r_1 = a_{n-2} - b_2 q_{n-3} - b_3 q_{n-4} - b_4 q_{n-5} - b_5 q_{n-6} \]

\[ r_2 = a_{n-1} - b_3 q_{n-3} - b_4 q_{n-4} - b_5 q_{n-5} \]

\[ r_3 = a_n - b_4 q_{n-3} - b_5 q_{n-4} \]

\[ r_4 = a_{n+1} + b_5 q_{n-3} \]

where particularly for \( r_1, r_2, r_3 \) and \( r_4 \), \( n \) is the degree \( (P) \)

**Illustrative Example**

Divide \( x^7 - 3x^6 - 14x^5 - 4x^4 + 27x^3 + 15x^2 - 7x - 6 \) by \( x^3 + 2x^2 - x - 4 \).

**Solution:**

Step 1: Use \( n \), the degree \( (P) \) and \( m \), the degree \( (D) \) to set the equations of the quotient \( Q(x) \) and the remainder \( R(x) \).

Given \( n = 7 \) and \( m = 3 \). Then

\[ Q(x) = q_1 x^4 + q_2 x^3 + q_3 x^2 + q_4 x + q_5 \]

and

\[ R(x) = r_1 x^2 + r_2 x + r_3 \]

Step 2: Refer from \( P(x) \) and \( D(x) \) and list the values of \( a_1, a_2, ..., a_{n+1} \) and \( b_1, b_2, ..., b_{m+1} \) respectively.

\[
\begin{align*}
  a_1 &= 1 \\
  a_2 &= -3 \\
  a_3 &= -14 \\
  a_4 &= -4 \\
  a_5 &= 27 \\
  a_6 &= 15 \\
  a_n &= a_7 = -7 \\
  a_{n+1} &= a_8 = -6 \\
  b_1 &= 1 \\
  b_2 &= 2 \\
  b_3 &= -1 \\
  b_4 &= -4 \\
  b_m &= b_3 = -1 \\
  b_{m+1} &= b_4 = -4 \\
\end{align*}
\]

Step 3: Evaluate the coefficients of \( Q(x) \) and \( R(x) \) by applying the respective formula for \( q_1, q_2, ..., q_5, r_1, r_2, r_3 \) and \( r_4 \), and substituting the values of \( a_1, a_2, ..., a_8 \) and \( b_1, b_2, ..., b_4 \).

\[ q_1 = a_1 = 1, \]
\[ q_2 = a_2 - b_2 q_1 = -3 - (2)1 = -5 \]
\[ q_3 = a_3 - b_2 q_2 - b_3 q_1 = -14 - (2)(-5) - (1)(1) = -3 \]
\[ q_4 = a_4 - b_2 q_3 - b_3 q_2 - b_4 q_1 = -4 - (2)(-3) - (1)(-5) - (4)(1) = 1 \]
\[ q_5 = a_5 - b_2 q_4 - b_3 q_3 - b_4 q_2 = 27 - (2)(1) - (1)(-3) - (4)(-5) = 2 \]

while

\[ r_1 = a_{n-1} - b_2 q_{n-2} - b_3 q_{n-3} - b_4 q_{n-4} = a_6 - b_2 q_5 - b_3 q_4 - b_4 q_3 = 15 - (2)(2) = (1)(1) = -4(3) = 0 \]

\[ r_2 = a_n - b_3 q_{n-2} - b_4 q_{n-3} = a_7 - b_3 q_5 - b_4 q_4 = -7 - (1)(2) = (4)(1) = -1 \]

\[ r_3 = a_{n+1} - b_4 q_{n-2} = a_8 - b_4 q_5 = -6 - (4)(2) = 2 \]

Thus, the quotient \( Q(x) \) and the remainder \( R(x) \) when \( x^7 - 3x^6 - 14x^5 - 4x^4 + 27x^3 + 15x^2 - 7x - 6 \) is divided by \( x^3 + 2x^2 - x - 4 \), \( Q(x) = 1x^4 - 5x^3 - 3x^2 + x + 2 = x^4 - 5x^3 - 3x^2 + x + 2 \) and

\[ R(x) = 0x^2 - 1x + 2 = -x + 2 \]

To verify the result using long division,

\[
\frac{x^4 - 5x^3 - 3x^2 + x + 2}{x^3 + 2x^2 - x - 4} = \frac{x^7 - 3x^6 - 14x^5 - 4x^4 + 27x^3 + 15x^2 - 7x - 6}{x^7 + 2x^6 - x^5 - 4x^4 - 5x^3 - 10x^2 + 5x + 20x + 20x^2 + 3x^3 + 15x^2 + 3x^2 + 12x^2 + 7x + 2x^3 + 4x^2 - x^2 - 4x - 2x - 8 - x + 2}
\]

And by compact synthetic division, the coefficients of the divisor are -2, 1, 4 while those
of the dividend are 1, -3, -14, -4, 27, 15, -7, -6. Then

\[
\begin{array}{c|cccccccc}
\text{q} & 1 & 4 & 6 & 10 & 13 & 15 & 18 & 20 & 22 \\
\hline
b & -2 & -3 & -14 & -4 & 27 & 15 & -7 & -6 & -2
\end{array}
\]

From these resulting coefficients, the quotient is \(x^4 - 5x^3 - 3x^2 + x + 2\) and the remainder is \(-x + 2\).

Thus, long division and compact synthetic division also show that

\[
\frac{x^7 - 3x^6 - 14x^5 - 4x^4 + 27x^3 + 15x^2 - 7x - 6}{x^3 + 2x^2 - x - 4} = x^4 - 5x^3 - 3x^2 + x + 2 + \frac{x^2}{x^3 + 2x^2 - x - 4}
\]

When \(b_1 \neq 1\)

1. Use \(n\), the \(\deg(P)\) and \(m\), the \(\deg(D)\) to set the equations of the quotient \(Q(x)\) and the remainder \(R(x)\). To do this,

   a. if \(m = 2\), then
   \[
   Q(x) = q_1x^{n-2} + q_2x^{n-3} + q_3x^{n-4} + \ldots + q_{n-2}x + q_{n-1}
   \]
   \[
   R(x) = r_1x + r_2
   \]

   b. if \(m = 3\), then
   \[
   Q(x) = q_1x^{n-3} + q_2x^{n-4} + q_3x^{n-5} + \ldots + q_{n-3}x + q_{n-2}
   \]
   \[
   R(x) = r_1x^2 + r_2x + r_3
   \]

   c. if \(m = 4\), then
   \[
   Q(x) = q_1x^{n-4} + q_2x^{n-5} + q_3x^{n-6} + \ldots + q_{n-4}x + q_{n-5}
   \]
   \[
   R(x) = r_1x^3 + r_2x^2 + r_3x + r_4
   \]

2. Refer from \(P(x)\) and \(D(x)\) and list the values of \(a_1, a_2, \ldots, a_{n+1}\) and \(b_1, b_2, \ldots, b_{m+1}\) respectively.

3. Evaluate the coefficients of \(Q(x)\) and \(R(x)\) by applying the respective formulas for \(t_i\) and \(r_i\), and substituting the values of \(a_1, a_2, \ldots, a_{n+1}\) and \(b_1, b_2, \ldots, b_{m+1}\). That is, in the division of any polynomial

   a. by a quadratic divisor,
   \[
   q_n = \frac{a_n - b_2q_{n-1} - b_3q_{n-2}}{b_1}
   \]
   while
   \[
   r_1 = a_n - b_2q_{n-1} - b_3q_{n-2}
   \]
   \[
   r_2 = a_{n+1} - b_3q_{n-1}
   \]
   where particularly for \(r_1\) and \(r_2\), \(n\) is the \(\deg(P)\);

   b. by a cubic divisor,
   \[
   q_n = \frac{a_n - b_2q_{n-1} - b_3q_{n-2} - b_4q_{n-3}}{b_1}
   \]
   while
   \[
   r_1 = a_{n-1} - b_2q_{n-2} - b_3q_{n-3} - b_4q_{n-4}
   \]
   \[
   r_2 = a_n - b_3q_{n-2} - b_4q_{n-3}
   \]
   \[
   r_3 = a_{n+1} - b_4q_{n-2}
   \]
   here particularly for \(r_1, r_2\) and \(r_3, n\) is the \(\deg(P)\); and

   c. by a quartic divisor,
   \[
   q_n = \frac{a_n - b_2q_{n-1} - b_3q_{n-2} - b_4q_{n-3} - b_5q_{n-4}}{b_1}
   \]
   where \(n \in \mathbb{Z}^+\)

   while
   \[
   r_1 = a_{n-2} - b_2q_{n-3} - b_3q_{n-4} - b_4q_{n-5} - b_5q_{n-6}
   \]
   \[
   r_2 = a_{n-1} - b_3q_{n-3} - b_4q_{n-4} - b_5q_{n-5}
   \]
   \[
   r_3 = a_n - b_4q_{n-3} - b_5q_{n-4}
   \]
   \[
   r_4 = a_{n+1} + b_5q_{n-3}
   \]
   where particularly for \(r_1, r_2, r_3\) and \(r_4, n\) is the \(\deg(P)\)

Illustrative Example:

Divide \(12x^{10} - 2x^7 + 15x^6 + 2x^4 - 6x^3 + 3x^2 - 2x + 6\) by \(3x^4 - 2x + 3\).

Solution

Step 1: Use \(n\), the \(\deg(P)\) and \(m\), the \(\deg(D)\), to set the equations of the quotient \(Q(x)\) and the remainder \(R(x)\).

Given \(n = 10\) and \(m = 4\).
Then
\[ Q(x) = q_3x^6 + q_2x^5 + q_3x^4 + q_4x^3 + q_5x^2 + q_6x + q_7 \]
and
\[ R(x) = r_1x^3 + r_2x^2 + r_3x + r_4 \]

Step 2: Refer from \( P(x) \) and \( D(x) \) and list the values of \( a_1, a_2, \ldots, a_{n+1} \) and \( b_1, b_2, \ldots, b_{m+1} \) respectively.

\[
\begin{align*}
    a_1 &= 12 & b_1 &= 3 \\
    a_2 &= 0 & b_2 &= 0 \\
    a_3 &= 0 & b_3 &= 0 \\
    a_4 &= -2 & b_4 &= -2 \\
    a_5 &= 15 & b_{m+1} &= b_5 = 3 \\
    a_6 &= 0 & b_7 &= 2 \\
    a_7 &= 2 & b_8 &= -6 \\
    a_8 &= 3 & b_9 &= 3 \\
    a_n &= a_{10} & = -2 \\
    a_{n+1} &= a_{11} & = 6
\end{align*}
\]

Step 3: Evaluate the coefficients of \( Q(x) \) and \( R(x) \) by applying the respective formula for \( q_1, q_2, \ldots, q_7, r_1, r_2, \ldots, r_4 \), and substituting the values of \( a_1, a_2, \ldots, a_{11} \) and \( b_1, b_2, \ldots, b_5 \).

\[
\begin{align*}
    q_1 &= \frac{a_1}{b_1} = 12 \cdot 3 = 4 \\
    q_2 &= \frac{a_2 - b_1q_1}{b_1} = \frac{0 - 0(4)}{3} = 0 \\
    q_3 &= \frac{a_3 - b_1q_2 - b_2q_1}{b_1} = \frac{0 - 0(0)(0) - (0)(0)}{3} = 0 \\
    q_4 &= \frac{a_4 - b_1q_3 - b_2q_2 - b_3q_1}{b_1} = \frac{-2 - (0)(0)(0) - (0)(0)}{3} = 2 \\
    q_5 &= \frac{a_5 - b_2q_4 - b_3q_3 - b_4q_2 - b_5q_1}{b_1} \\
        &= \frac{15 - (0)(2)(0) - (0)(0) - (0)(3)(0)}{3} = 1 \\
    q_6 &= \frac{a_6 - b_2q_5 - b_3q_4 - b_4q_3 - b_5q_2}{b_1} \\
        &= \frac{0 - (0)(1) - (0)(2) - (0)(0)(3)(0)}{3} = 0 \\
    q_7 &= \frac{a_7 - b_2q_6 - b_3q_5 - b_4q_4 - b_5q_3}{b_1} \\
        &= \frac{2 - (0)(0)(0) - (0)(1)(-2)(0) - (0)(3)(0)}{3} = 2
\end{align*}
\]

while

\[
\begin{align*}
    r_1 &= a_{n-2} - b_2q_{n-3} - b_3q_{n-4} - b_4q_{n-5} - b_5q_{n-6} = a_8 - b_2q_7 - b_3q_6 - b_4q_5 - b_5q_4 = -6 - 0(7) - (0)(0) - (2)(1) - (3)(2) = -6 \\
    r_2 &= a_{n-1} - b_2q_{n-2} - b_3q_{n-4} - b_5q_{n-6} = a_9 - b_2q_7 - b_3q_6 - b_5q_4 = 3 - 0(2) - (0)(0) - (3)(1) = 0 \\
    r_3 &= a_n - b_4q_{n-3} - b_5q_{n-4} = a_{10} - b_4q_7 - b_5q_6 = 0 - 0(2) - (3)(0) = 0 \\
    r_4 &= a_{n+1} + b_5q_{n-3} = a_{11} + b_5q_7 = 6 - (3)(2) = 0
\end{align*}
\]

Thus, the quotient \( Q(x) \) and the remainder \( R(x) \) when \( 12x^{10} - 2x^7 + 15x^6 + 2x^4 + 6x^3 + 3x^2 - 2x + 6 \) is divided by \( 3x^4 - 2x + 3 \) are
\[
Q(x) = 4x^6 + 6x^5 + 6x^4 + 2x^3 + 3x^2 + 2x + 2
\]
and
\[
R(x) = -10x^3 + 2x^2 + 2x + 0 = -10x^3 + 2x
\]

To verify the result using long division,
\[
\begin{array}{cccccccccccccc}
 & & 4 & + & 6 & + & 2 & & \\
\hline
3x^4 & + & 0x^3 & + & 3x^2 & + & 2x & + & 0
\end{array}
\]

And by compact synthetic division, the divisor should be written first as \( x^4 + 0x^3 + 0x^2 - \frac{2}{3}x + 1 \) so that the coefficients to be used are \( 0, \frac{2}{3}, -1 \) while the coefficients of the dividend are \( 12, 0, 0, -2, 15, 0, 2, -6, 3, -2, 6 \). Then,

\[
\begin{array}{cccccccccccccc}
& & 2 & + & 3 & - & 1 & & & & & & & & \\
& & 12x^6 & + & 6x^5 & + & 3x^4 & + & 6x^3 & + & 3x^2 & + & 6x & + & 2x & + & 0
\end{array}
\]

From these resulting coefficients, the quotient is \( \frac{1}{3}(12x^6 + 6x^5 + 3x^4 + 6x^3 + 3x^2 + 6) = 4x^6 + 2x^3 + x^2 + 2 \) and the remainder is \( -10x^3 + 2x \).
Thus, long division and compact synthetic division also show that
\[
\begin{align*}
12x^{10} - 2x^7 + 15x^6 + 2x^5 - 6x^4 + 3x^2 - 2x + 6 &= 3x^4 - 2x + 3 \\
= 4x^6 + 2x^3 + x^2 + 2 + \frac{-10x^3 + 2x}{3x^4 - 2x + 3}
\end{align*}
\]

Conditions to the Problems where the Usual Synthetic Division is Inappropriate

The usual synthetic division is applied only to the division of polynomials with linear divisor \(x - c\), where \(c\) is a constant. This is basically inappropriate to problems whose divisor is in higher degrees. With this, the following are the additional conditions for problems where the usual synthetic division is inappropriate aside from being \(\deg(D) > 1\).

1. The divisor \(D(x) = b_1x^m + b_2x^{m-1} + b_3x^{m-2} + \ldots + b_mx + b_{m-1}\) is prime or irreducible.
2. In case \(D(x)\) is reducible, at least one of the factors of \(D(x)\) is irreducible non-linear polynomial.
3. In case \(D(x)\) is reducible and its factors are all linear polynomial (either monic or non-monic), one of them is not a factor of \(P(x)\).

To illustrate the first condition where the divisor is irreducible, consider the following. Explain why usual synthetic division is inappropriate for \(\frac{x^5 + 2x^3 - x^2 - 2}{x^2 + 2}\).

Solution:
The divisor \(x^2 + 2\) is irreducible. Thus, the usual synthetic division is inappropriate for \(\frac{x^5 + 2x^3 - x^2 - 2}{x^2 + 2}\) based on Condition (1).

To illustrate the second condition where “in case the divisor is reducible, at least one of the factors of \(D(x)\) is irreducible non-linear polynomial”, consider the following. Verify if usual synthetic division is appropriate for \(\frac{x^6 + 6x^5 - 2x^4 - 60x^3 - 71x^2 + 54x + 72}{x^3 + x^2 + 7x + 20}\).

Solution:
The divisor \(x^3 + x^2 - 7x + 20\) is reducible and can be written as \((x^2 - 3x + 5)\)(x + 4). But the factor \((x^2 - 3x + 5)\) is an irreducible non-linear polynomial. Thus, the usual synthetic division is inappropriate for \(\frac{x^6 + 6x^5 - 2x^4 - 60x^3 - 71x^2 + 54x + 72}{x^3 + x^2 + 7x + 20}\) based on Condition (2).

To illustrate the third condition where “in case the divisor \(D(x)\) is reducible and its factors are all linear polynomial (either monic or non-monic), one of them is not a factor of \(P(x)\)”, consider the following. Show that usual synthetic division is inappropriate for \(\frac{x^6 - 3x^5 - 17x^4 + 39x^3 + 88x^2 - 108x - 144}{x^4 - 15x^2 - 10x + 24}\).

Solution:
The divisor \(x^4 - 15x^2 - 10x + 24\) is reducible and can be written as product of linear polynomials, that is, \((x + 3)(x + 2)(x - 1)(x - 4)\). By applying Factor Theorem with \(P(x) = x^6 - 3x^5 - 17x^4 + 39x^3 + 88x^2 - 108x - 144\) at \(x = -3, -2, 1\) and \(4\), \(P(-3) = 0\), \(P(-2) = 0\), \(P(1) = -144\) and \(P(4) = 0\). This means \((x + 3), (x + 2)\) and \((x - 4)\) are factors while \((x - 1)\) is not a factor of \(P(x)\). Since one of the factors of \(D(x)\) is not a factor of \(P(x)\), then the usual synthetic division is still inappropriate for \(\frac{x^6 + 6x^5 - 2x^4 - 60x^3 - 71x^2 + 54x + 72}{x^3 + x^2 + 7x + 20}\) based on Condition (3).

And to illustrate a problem that are not within the three conditions, consider the following. Verify if usual synthetic division is appropriate for \(\frac{x^3 - 3x^2 - 10x + 24}{x^2 - 3x^2 - 10x + 24}\).

Solution:
The divisor \(x^3 - 3x^2 - 10x + 24\) is reducible and can be written as product of linear polynomials \((x - 4)(x - 2)(x + 3)\). By applying Factor Theorem with \(P(x) = x^3 - 3x^2 - 10x + 24\) at \(x = 4, 2\) and \(-3\), \(P(4) = 0\), \(P(2) = 0\) and \(P(-3) = 0\). This means \((x - 4), (x - 2)\) and \((x + 3)\) are factors of \(P(x)\). Since this problem is not within the three conditions, then the series of usual synthetic divisions is appropriate for \(\frac{x^3 - 3x^2 - 10x + 24}{x^3 - 3x^2 - 10x + 24}\).
To sum up, three conditions were identified in the divisor $D(x)$ of higher degrees relative to the inappropriateness of usual synthetic division to $\frac{P(x)}{D(x)}$. First, if $D(x)$ is irreducible. That is, if the divisor cannot be written as a product of polynomials of lower degrees. Second, if $D(x)$ is reducible but still there exists at least one non-linear irreducible factor. And third, if $D(x)$ is reducible and all factors are linear polynomials but one of these factors of $D(x)$ is not a factor of $P(x)$.

Therefore, the conditions have something to do with the reducibility of $D(x)$ and nature of the linear polynomial factor with respect to $P(x)$.

**Generalized Algorithm of Dividing Polynomials of Higher Degrees**

In the first part of this paper, the developed algorithms and formulas were presented and were verified applicable for the divisions of polynomials with quadratic, cubic and quartic divisors. In problems involving these divisions of polynomials with quadratic, cubic and quartic divisors, the developed algorithm is presented as an alternative solution.

Now, a generalized algorithm is also presented. Given the dividend $P(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \cdots + a_nx + a_{n+1}$ and the divisor $D(x) = b_1x^m + b_2x^{m-1} + b_3x^{m-2} + \cdots + b_mx + b_{m+1}$ where $n > m$, $a_1 \neq 0$, $b_1 \neq 0$, and $a_i$’s and $b_i$’s including $a_{n+1}$ and $b_{m+1}$ are constant,

1. Use $n$, the deg($P$) and $m$, the deg($D$), to set the equations of the quotient $Q(x)$ in the form
   
   $Q(x) = q_1x^{n-m} + q_2x^{n-m-1} + q_3x^{n-m-2} + \cdots + q_{n-m}x + q_{n-m+1}$

   and the remainder $R(x)$ in the form
   
   $R(x) = r_1x^{m-1} + \cdots + r_{m-3}x^3 + r_{m-2}x^2 + r_{m-1}x + r_m$

   where $q_{n-m+1}$ and $r_m$ are the constant terms.

2. Use the general formula $q_n = a_n - b_2q_{n-1} - \cdots - b_{m+1}q_{n-m}$ if $b_1 = 1$ or $q_n = \frac{a_n-b_2q_{n-1}-\cdots-b_{m+1}q_{n-m}}{b_1}$ if $b_1 \neq 1$, where $n \in \mathbb{Z}^+$ and $m = \text{deg}(D)$, to generate the formula of the needed $q_1, q_2, \ldots, q_{n-m+1}$ and $r_1, r_2, \ldots, r_m$.

Take note of the $a_i$’s, $b_i$’s and $q_i$’s that should only be involved in generating the formulas.

3. Evaluate $q_1, q_2, \ldots, q_{n-m+1}$ and $r_1, r_2, \ldots, r_m$ using the values of $a_1, a_2, \ldots, a_{n+1}$ from $P(x)$ and $b_1, b_2, \ldots, b_{m+1}$ from $D(x)$.

It is important to note that writing the generalized formulas for the coefficients of both quotient and remainder is the important part or step of the algorithm. That is, aside from quadratic, cubic and quartic polynomial divisors, divisions with other non-linear divisors in higher degrees can be done. In deriving such generalized formulas, the following observations from previous illustrations were noted and considered:

1. Patterns show that the coefficients of the quotient $q_n$ can be extended when needed as shown:
   
   $q_n = a_n - b_2q_{n-1} - b_3q_{n-2} - \cdots - b_{m+1}q_{n-m}$,

   where $b_{m+1}q_{n-m}$ is the last possible part of the formula.

2. The degree of the quotient $Q(x)$ is in ‘degree of the dividend less the degree of the divisor’. This means the leading term of $Q(x)$ is $q_1x^{n-m}$.

3. The remainder $R(x)$ is in the degree ‘one less than the degree of the divisor’. This means the leading term of $R(x)$ is $q_1x^{m-1}$.

4. The coefficients $r_1$ of the remainder $R(x)$ also have the following equivalent formulas in terms of $q_i$: $r_1 = q_{n-m+2}$, $r_2 = q_{n-m+3}$, $r_3 = q_{n-m+4}$, and so on, up to the constant term $r_m$. It should be noted that the values of $r_i$ does not need multiplication by $\frac{1}{b_1}$ even the divisor $D(x)$ is not monic, that is, $b_1 \neq 1$.

5. In writing or generating formula of $q_i$ in terms of $a_i$ and $b_i$, the coefficients that involve $a_i$ for $1 \leq i \leq n + 1$ and $b_i$ for $2 \leq i \leq m + 1$ shall only be used as parts of the formulas. Relatively, in writing or generating formula of $r_i$ in terms to $b_i$ and $q_i$, the coefficients that involve $b_i$ for $2 \leq i \leq m + 1$ and $q_i$ for $1 \leq i \leq (n - m + 1)$ shall only be used as parts of the formulas. All other coefficients that are not included in the
respective given ranges and that may appear in generating the formulas shall be deemed not necessary or non-existing in the formula of \( q_i \) and \( r_i \).

There are apparently challenges that can be encountered in applying the developed algorithm. Familiarizing the patterns using the generalized formula may be a difficulty on generating the formulas for the coefficients of the quotient \( (q_i) \) and the remainder \( (r_i) \). Also, this algorithm requires considerations particularly careful identification and elimination of the unnecessary part or parts of the formulas for \( q_i \) and \( r_i \). Nevertheless, the developed algorithm can be used as an alternative method of dividing polynomials of higher degrees that will only require substitution of coefficients of the dividend and the divisor to get \( q_i \) and \( r_i \). This will no longer require the use of variables as used in long division and piling of coefficients as parts of the steps in compact synthetic division.

Moreover, Table 1 exhibits the summary of the limitations of usual synthetic division and the applicability of the developed algorithm as an alternative solution.

**Illustrative Example.**

Divide \( 3x^2 + x^6 - 10x^5 - 2x^4 - x^3 - 21x^2 - 6x + 8 \) by \( 3x^3 + 4x^2 - 3x - 4 \).

**Solution:**

The divisor \( 3x^3 + 4x^2 - 3x - 4 \) is reducible and can be written as product of linear polynomials, that is, \( (3x + 4)(x + 1)(x - 1) \). By applying Factor Theorem with \( P(x) = 3x^3 + 4x^2 - 3x - 4 \) at \( x = \pm\frac{4}{3}, -1 \) and 1, \( P\left(\frac{4}{3}\right) = 0, P(-1) = 0 \) and \( P(1) = -28 \).

This means \((3x + 4) \) and \((x + 1)\) are factors while \((x - 1)\) is not factor of \( P(x) \). Since one of the factors of \( D(x) \) is not a factor of \( P(x) \), then the usual synthetic division is still inappropriate for \( \frac{3x^7 + x^6 - 10x^5 - 2x^4 - x^3 - 21x^2 - 6x + 8}{3x^3 + 4x^2 - 3x - 4} \) based on Condition (c) and series of this division will result to incorrect quotient.

But as an alternative solution, this can be solved using the generalized algorithm with the following steps:

**Step 1:** Use \( n, \deg(P) \) and \( m, \deg(D) \), to set the equations of the quotient \( Q(x) \) and the remainder \( R(x) \)

\[ Q(x) = q_1x^4 + q_2x^3 + q_3x^2 + q_4x + q_5 \quad \text{and} \quad R(x) = r_1x^2 + r_2x + r_3 \]

Then

\[ Q(x) = q_1x^4 + q_2x^3 + q_3x^2 + q_4x + q_5 \quad \text{and} \quad R(x) = r_1x^2 + r_2x + r_3 \]

**Step 2:** Since \( b_1 = 3 \), then use the general formula

\[ q_n = \frac{a_n - b_2q_{n-1} - b_3q_{n-2} - \cdots - b_{m+1}q_{n-m}}{b_1} \]

to generate the formula of \( q_1, q_2, ..., q_5 \). Take note that formulas should only involve \( a_1, a_2, ..., a_8, b_1, b_2, b_4 \) and \( q_1, q_2, ..., q_5 \).

\[ q_1 = \frac{a_1}{b_1} \]
Step 3: Evaluate $q_1, q_2, \ldots, q_5$ and $r_1, r_2, \ldots, r_3$ using the values of $a_1, a_2, \ldots, a_6$ from $P(x)$ and $b_1, b_2, \ldots, b_4$ from $D(x)$.

\[
\begin{align*}
q_1 &= \frac{a_1}{b_1} = \frac{3}{3} = 1 \\
q_2 &= \frac{a_2 - b_2 q_1}{b_1} = \frac{1 - (4)(1)}{3} = -\frac{3}{3} = -1 \\
q_3 &= \frac{a_3 - b_3 q_2 - b_3 q_1}{b_1} = \frac{-10 - (4)(-1) - (-3)(1)}{3} = -\frac{3}{3} = -1 \\
q_4 &= \frac{a_4 - b_4 q_3 - b_3 q_2 - b_4 q_1}{b_1} = \frac{-2 - (4)(-1) - (-3)(-1) - (-4)(1)}{3} = \frac{3}{3} = 1 \\
q_5 &= \frac{a_5 - b_5 q_4 - b_4 q_3 - b_3 q_2 - b_4 q_1}{b_1} = \frac{-1 - (4)(1) - (-3)(-1) - (-4)(-1)}{3} = -\frac{12}{3} = -4
\end{align*}
\]

while
\[
\begin{align*}
r_1 &= -21 - (4)(-4) - (-3)(1) - (-4)(-1) = -6 \\
r_2 &= a_7 - b_3 q_5 - b_4 q_4 = -6 - (-3)(-4) - (-4)(1) = 14 \\
r_3 &= a_8 - b_4 q_5 = 8 - (-4)(-4) = 8 \\
\end{align*}
\]

Therefore,
\[
\begin{align*}
3x^7 + x^6 - 10x^5 - 2x^4 - x^3 - 21x^2 - 6x + 8 \\
3x^3 + 4x^2 - 3x - 4 \\
= q_1 x^2 + q_2 x^3 + q_3 x^2 + q_4 x + q_5 \\
+ \frac{r_1 x^2 + r_2 x + r_3}{2x^3 - x^2 - 25x + 33}
\end{align*}
\]

To verify the result long division,
\[
\begin{align*}
3x^7 + x^6 - 10x^5 - 2x^4 - x^3 - 21x^2 - 6x + 8 \\
3x^3 + 4x^2 - 3x - 4 \\
= x^4 - x^3 - x^2 + x - 4 \\
+ \frac{6x^2 - 14x - 8}{2x^3 - x^2 - 25x + 33}
\end{align*}
\]

And by using compact synthetic division, the divisor should be written first as $x^3 + \frac{4}{3}x^2 - x - \frac{4}{3}$, so that the coefficients to be used are $-\frac{4}{3}, 1, \frac{4}{3}$ while the coefficients of the dividend are $3, 1, -10, -2, -1, -21, -6, 8$.

Then

\[
\begin{array}{c|cccc}
4 & 1 & 4 & 16 \\
3 & -3 & 3 & -12 \\
-3 & 1 & -10 & -6 & -8
\end{array}
\]
From these resulting coefficients, the quotient is \( \frac{1}{3} (3x^4 - 3x^3 - 3x^2 + 3x - 12) = x^4 - x^3 - x^2 + x - 4 \) and the remainder is \(-6x^2 - 14x - 8\).

Thus, long division and compact synthetic division also show that
\[
\begin{align*}
3x^7 + x^6 - 10x^5 - 2x^4 - x^3 - 21x^2 - 6x + 8 \\
3x^3 + 4x^2 - 3x - 4 \\
\frac{-6x^2 - 14x - 8}{2x^3 - x^2 - 25x + 33}
\end{align*}
\]

CONCLUSIONS

In light of the findings, the following conclusions are drawn:

1. Using the synthetic division in compact arrangement by quadratic divisor, cubic divisor and quartic divisor, algorithms showing procedure were developed when \( b_1 = 1 \) and \( b_1 \neq 1 \). These algorithms on divisions of any polynomial dividend by the divisors mentioned are then alternative solutions.
2. For the problems (division of polynomials with non-linear divisors) where usual synthetic divisions are inappropriate, additional conditions were provided, and can be verified through examples and counterexamples. The problems with these conditions may be solved using the developed algorithm.
3. The developed algorithm of dividing polynomials of higher degrees, that is, any polynomial dividend by any polynomial divisor, was generalized with considerations of some conditions and observations on the patterns. Illustrative examples using compact synthetic and long divisions of polynomials can be used to verify the result.

RECOMMENDATIONS

Considering the findings and conclusions in this paper, the following recommendations were then endorsed:

1. Like conditions on inapplicability of the usual synthetic division realized from this study, additional and related conditions even for the developed algorithm may be explored and provided.
2. Another alternative procedure in finding the quotient of polynomials in higher degrees may be developed.
3. Relative to this paper, further research may be conducted, for example, on efficiency of this algorithm in Mathematics classes.

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